Interaction of a Plasma with a "Helical" Electron Beam

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The behavior of a plasma-beam system immersed in a magnetic field $\mathbf{B}_0$ is investigated theoretically under the assumption that the electrons comprised in the beam have helical trajectories coaxial with $\mathbf{B}_0$. The beam is designated as "helical." The plasma-beam system is examined for instabilities which result in growing transverse waves aligned in a direction parallel to that of $\mathbf{B}_0$. Two classifications of instabilities are introduced. In accordance with one of these, the instabilities can be "superluminous," "subluminous," or "counterstreaming," depending on direction and velocity of the excited wave. The other classification differentiates between excited "*P*" and "*B*" waves and is based on the comparison of some of the waves excited by the beam and the corresponding waves which can be radiated by a single particle in the unperturbed plasma. If the beam is linear (not helical), i.e., if the electron trajectories in the beam are parallel to $\mathbf{B}_0$, there is only one type of instability ("*P*" instability) in which the frequencies of the waves excited by the beam differ very little from the frequency of the corresponding waves which can be radiated by a single particle in the undisturbed plasma. On the other hand, if the beam is helical there is an additional instability ("*B*" instability). The latter instability occupies a relatively wide frequency range in the neighborhood of the corresponding "*P*" instability and represents a continuation of the "*P*" instability along the frequency axis. When the intensity of the helical beam is sufficiently small, the "*P*" instability is "strong" and the "*B*" instability is "weak," i.e., the rate of growth of the "*B*" waves is relatively small. Physical conditions under which different types of instability may occur are discussed and the above types of plasma-beam instabilities are investigated with particular reference to the frequency ranges and rates of growth of the excited waves. It is shown that a "*P*" instability is always multiple, i.e., there are at least two excited waves having different frequencies which are aligned in the direction of the magnetic field. (A multiple "*P*" instability does not occur if the beam is linear.) The character of the multiple "*P*" instability is analyzed for various transverse and longitudal velocity components of the electrons in the beam. Using a graphical analysis of the dispersion equation for the plasma-beam system in the ω-k plane (where ω is the frequency and k is the wave number), it is found that superluminous and subluminous instabilities are convective, whereas counterstreaming instabilities are nonconvective.

INTRODUCTION

THIS investigation deals with the interaction of a stationary plasma and an electron beam in the presence of a static magnetic field $\mathbf{B}_0$. It is assumed that the electrons in the beam have helical trajectories coaxial with $\mathbf{B}_0$. Each of these electrons has a velocity component $v_{\parallel}=\beta_{\parallel}c$ in the direction of $\mathbf{B}_0$ and a velocity component $v_{\perp}=\beta_{\perp}c$ in the direction perpendicular to $\mathbf{B}_0$. The beam formed by these electrons is designated as "helical" and it moves with velocity $v_{\parallel}$ (or $\beta_{\parallel}$) in the direction parallel to that of $\mathbf{B}_0$. The helical beam can be expressed in the form of a distribution function in electron velocity space. Using rectangular coordinates v_x, v_y, v_z in which the v_z axis is aligned in the direction of $\mathbf{B}_0$ and assuming that

$$v_r^2=v_x^2+v_y^2 \tag{0.1}$$

one can represent such a distribution function as follows[1]:

$$f(v_z,v_r)=(1/2\pi v_r)\delta(v_z-v_{\parallel})\delta(v_r-v_{\perp}). \tag{0.2}$$

An analysis is given of instabilities which result from the interaction of a helical beam with a stationary cold plasma and give rise to transverse (circularly polarized) waves moving in the direction parallel to that of $\mathbf{B}_0$. Various types of plasma-beam systems are investigated with particular reference to the frequency ranges and rates of growth of the waves associated with these instabilities.

The interaction of a plasma with a helical beam is of relatively frequent occurrence in astrophysics, geophysics, thermonuclear research, etc. In the early stages of the development of radio astronomy it was suggested by Kiepenheuer[2] that the nonthermal radio emissions from the sun are due to beams of electrons produced at the sunspots and directed at an arbitrary angle to the magnetic field. These helical beams spiral out of the sunspots into the surrounding plasma which forms the solar corona. Similar types of interactions occur under terrestrial conditions in the outer regions of the ionosphere (exosphere). They are caused by helical beams which comprise electrons and protons gyrating around the earth's magnetic field and are believed to produce auroral effects. These beams interact with the plasma in the exosphere and generate very low frequency electromagnetic radiations which are detected at the earth's surface in the form of continuous noise and whistlers.[3] These radiations were described apparently for the first time by Menzel and Salisbury[4] and subsequently studied by others.

The results of this paper are expressed in terms of

[1] V. V. Zhelezniakov, Izv. Vysshikh Uchebn-Zavedenii Radiofiz. **111**, 57 (1960) [English transls.: News of Higher Educational Institutions, Ministry of Higher Education, U.S.S.R., Radio Physics Series, Vol. 3, No. 1 (1960); Joint Publications Research Service, No. 7377 (1961)].

[2] K. O. Kiepenheuer, Nature **158**, 340 (1946).

[3] L. R. O. Storey, Phil. Trans. Roy. Soc. London Ser. A **246**, 113 (1953).

[4] D. H. Menzel and W. W. Salisbury, Nature **161**, 91 (1948).

appropriate nondimensional parameters defining various types of plasma systems. This has been done in order to facilitate the applications of these results to representative types of plasma which occur in nature or which may be produced in a laboratory. A very useful classification of plasmas based on these nondimensional parameters was introduced by Denisse and Delcroix.[5] On the basis of the results obtained in this paper, various observable phenomena which occur in plasma-beam systems can be predicted.

Relatively few publications which have appeared to date deal with instabilities produced in a plasma by a helical beam. In the past most of the investigations were concerned with a beam in which the trajectories of the charged particles are aligned in the direction parallel to that of $\mathbf{B}_0$. A rather extensive bibliography dealing with plasma-beam instabilities is listed in a recent review paper by Fainberg.[6]

The most pertinent work dealing with the subject of this paper is that of Zhelezniakov.[1] Zhelezniakov formulated the dispersion equation for a system in which a helical beam interacts with a stationary plasma and obtained specific solutions of this equation on the assumption that the stationary plasma is characterized by a refractive index $N(\omega)$ (where ω is the frequency of the waves resulting from the interaction). The present investigation provides an analysis of various aspects of the interaction which were not considered by Zhelezniakov, takes into account the motions of both ions and electrons in the stationary plasma and points out the significance of various physical factors which determine the character of the instabilities.

In the following discussion two classifications of instabilities are introduced. The first classification is based on the comparison of an excited wave in the plasma-beam system with a stationary wave which may be radiated by a single particle in the undisturbed plasma. This classification deals with two types of instabilities which will be designated, respectively, as a "P" instability and a "B" instability. In a P instability a growing wave (P wave) has the property that for decreasing beam intensity the frequency of this wave approaches as a limit the frequency of a wave radiated by a single particle in the undisturbed plasma. If the beam is linear, i.e., if the electrons in the beam have trajectories aligned along the direction parallel to that of $\mathbf{B}_0$ ($\beta_\perp=0$), there is only one type[7] of instability (P instability). On the other hand, if the beam is helical ($\beta_\perp\neq0$), there is an additional instability (B instability) which gives rise to excited B waves. A B instability occupies a relatively wide frequency range in the neighborhood of a P instability and represents a continuation of the P instability along the frequency axis.

If the intensity of the helical beam is sufficiently small, the P instability is "strong" and the B instability is "weak," i.e., the rate of growth of P waves is relatively large, whereas the rate of growth of B waves is relatively small.[8]

The second classification of plasma-beam instabilities is based on the relationship between the velocity of the beam and the phase velocity of the excited waves. In accordance with the second classification, the instabilities will be labeled as "superluminous," "subluminous," or "counterstreaming." In a superluminous instability the velocity of the beam exceeds the phase velocity of the excited wave; in a subluminous instability it is less than the phase velocity of the excited wave; and in both cases the direction in which the excited wave moves is the same as that of the beam. In a counterstreaming instability, the excited wave moves in the direction opposite to that of the beam.

In recent literature dealing with instabilities produced by a beam, one can find several references containing a discussion of superluminous and subluminous instabilities associated with transverse waves aligned in the direction parallel to that of the externally applied magnetic field. The occurrence of these instabilities depends on the character of the beam. If the beam is linear, i.e., if $\beta_\perp=0$, the instability is always superluminous.[9] For a helical beam, i.e., when $\beta_\perp\neq0$, there is a superluminous and subluminous instability.[10] It is pointed out in this investigation that in the case of a helical beam there is also a counterstreaming instability, and various properties of the superluminous, subluminous, and counterstreaming instabilities are investigated in detail.

It has also been found in this investigation that a P instability is always multiple, i.e., there are either two or four excited transverse waves having different frequencies which are aligned in the direction of the beam. If there are two excited waves, one of these represents a superluminous and the other a counterstreaming instability. If there are four excited waves, one represents a superluminous instability, another represents a counterstreaming instability, and the two remaining waves represent a subluminous instability.

A multiple P instability occurs only if the electron

[5] See J. F. Denisse and J. L. Delcroix, *Theorie des ondes dans les plasmas* (Dunod Cie., Paris, 1961), in particular Table III, 1 on p. 32. See also the equations (3.1) and (3.2) of this investigation which define the nondimensional parameters A and η characterizing various types of plasma. The quantity "A" used by Denisse and Delcroix is approximately the quantity "A^2" defined in (3.1).

[6] IA. B. Fainberg, J. Nucl. Energy: P. C **4**, 203 (1962).

[7] Jacob Neufeld and Harvel Wright, Phys. Rev. **129**, 1489 (1963).

[8] Jacob Neufeld, Phys. Rev. **124**, 1 (1961); see also Jacob Neufeld and P. H. Doyle, *ibid.* **121**, 654 (1961); and Jacob Neufeld, *ibid.* **127**, 346 (1962).

[9] The superluminous instability is directly related to the anomalous Doppler effect produced by oscillators moving with superluminous velocity in the direction of the magnetic field. See in that connection V. V. Zhelezniakov, Ref. 2, and G. G. Getmantzev and V. O. Rapoport, Zh. Eksperim. i Teor. Fiz. **38**, 1205 (1960) [English transl.: Soviet Phys.—JETP **11**, 871 (1960)].

[10] V. L. Zhelezniakov, Ref. 2, and IA. B. Fainberg, Ref. 1; See also a discussion on a related subject by A. A. Gaponov, Zh. Eksperim. i Teor. Fiz. **39**, 326 (1960) [English transl.: Soviet Phys.—JETP **12**, 232 (1961)].

beam is helical, i.e., a multiple instability does not occur if the electron beam is linear.

A discussion is given on the relative rates of growth of the excited waves representing a multiple P instability. The study of the relative rates of growth is useful in ascertaining which one of the multiple occurring instabilities is "dominant," i.e., has the largest rate of growth.

Analyzing the dispersion equation for the plasma-beam system in the $\omega-k$ plane,[11] one can ascertain which instabilities formulated in this paper are convective and which are nonconvective. In a convective instability a disturbance increases as it is carried along the system and it remains finite at each point. In a nonconvective instability a disturbance which originates in a limited region of space at any instance of time grows indefinitely for $t\to\infty$ in this region (in the linear approximation). It is found in this investigation that subluminous and superluminous instabilities are convective, whereas counterstreaming instabilities are nonconvective.

I. DISPERSION EQUATION

A. Formulation of the Problem

1. General

The problem outlined in the Introduction will be investigated under very simplified physical assumptions. That is, such factors as the presence of boundaries, density gradients, and temperature effects in the stationary plasma are not considered.

The stationary plasma is cold and uniform. It is dissipationless, extended in space, and contains no sources of energy. The electrons and ions in the stationary plasma have densities $(1-\sigma)n$ and n, respectively, and the density of the electrons in the beam is σn. Thus, the plasma-beam is charge equilibrated. It is assumed that the electron density of the beam is small when compared to the electron density of the stationary plasma, i.e.,

$$\sigma\ll 1. \tag{1.1}$$

The beam is considered to produce a small perturbation in the stationary plasma. Since $\beta_\perp\neq 0$, the beam is helical and moves with velocity $\beta_{\shortparallel}$. When $\beta_{\shortparallel}$ is positive (or negative) the beam moves in the direction of (or opposite to that of) $\mathbf{B}_0$.

The following symbols are used:

$$\omega_e=\left(\frac{4\pi ne^2}{m}\right)^{1/2},\ \omega_i=\left(\frac{4\pi ne^2}{M_i}\right)^{1/2};\ \Omega_e=\frac{|e|B_0}{mc};\ \Omega_i=\frac{|e|B_0}{M_ic}, \tag{1.2}$$

where m and e designate, respectively, the mass and the charge of an electron, and M_i is the mass of an ion in the stationary plasma. The ions are assumed to be singly charged. In the absence of the beam, i.e., when $\sigma=0$, the quantity n represents the electron and ion density in the unperturbed plasma. Consequently, ω_e is the plasma frequency. The quantities Ω_e and Ω_i represent, respectively, the magnitudes of the electron and ion gyrofrequencies.

2. Dispersion Equation

Consider a plasma-beam instability which results in the excitation of waves aligned in the direction parallel to that of $\mathbf{B}_0$, i.e., assume that

$$\mathbf{k}\|\mathbf{B}_0, \tag{1.3}$$

where $\mathbf{k}$ is the wave vector characterizing these waves. The waves defined by (1.3) may be longitudinal or transverse. The present investigation deals with circularly polarized transverse waves. The electrical intensity $\mathbf{E}$ of the excited waves is perpendicular to the wave vector $\mathbf{k}$ and rotates with the angular frequency ω. The dispersion equation for such waves has been formulated by Zheleznyakov[1] and is as follows:

$$F_p-\sigma F_b=0, \tag{1.4}$$

where

$$F_p\equiv F_p(\omega,k)=\omega^2-c^2k^2-\frac{\omega_i^2\omega}{\omega+\Omega_i}-\frac{(1-\sigma)\omega_e^2\omega}{\omega-\Omega_e} \tag{1.5}$$

and

$$F_b\equiv F_b(\omega,k)=\omega_e^2(1-\beta^2)^{1/2}\left[\frac{\omega-ck\beta_{\shortparallel}}{\omega-ck\beta_{\shortparallel}-\Omega_e(1-\beta^2)^{1/2}}+\frac{\beta_\perp^2(c^2k^2-\omega^2)}{2[\omega-ck\beta_{\shortparallel}-\Omega_e(1-\beta^2)^{1/2}]^2}\right]. \tag{1.6}$$

Since $\sigma\ll 1$, the expression $(1-\sigma)$ in the last term of the right-hand side of (1.5) will be replaced by 1. The quantity β^2 in (1.6) is as follows:

$$\beta^2=\beta_{\shortparallel}^2+\beta_\perp^2. \tag{1.7}$$

The expression F_p depends on the parameters of the stationary plasma, and

$$F_p=0 \tag{1.8}$$

represents the dispersion equation for the waves which can be transmitted through the stationary plasma (in the absence of the beam). The expression $-\sigma F_b$ represents the perturbation produced by the beam. For $\beta_\perp=0$ the equation (1.4) describes a system in which the electrons in the beam have trajectories aligned in the direction of $\mathbf{B}_0$.

A related problem dealing with transverse instabilities in a thermal plasma immersed in a magnetic field was investigated by Harris.[12] Harris considered a problem in which the "longitudinal" and "transverse" velocity spreads are different and obtained a dispersion

[11] P. A. Sturrock, Phys. Rev. **112**, 1988 (1958); see also, L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Inc., London, 1959), pp. 111–114.

[12] E. G. Harris, J. Nucl. Energy: P. C **2**, 138 (1961).

equation applicable to a nonrelativistic case which has to some extent a formal similarity to the one obtained by Zhelezniakov. One can also mention a related problem dealing with anisotropic velocity distributions discussed by Weibel.[13]

Following the customary procedure,[14] Eq. (1.4) is solved for ω assuming that k is real. The roots of (1.4) are represented in the form

$$\omega=\tilde{\omega}+\delta, \tag{1.9}$$

where $\tilde{\omega}$ is real and

$$\lim_{\sigma\to 0}\delta=0. \tag{1.10}$$

The roots in ω may be complex and if $\mathrm{Im}\delta<0$ there is an instability which is represented by an oscillation in which the amplitude grows exponentially with time. The quantity $\mathrm{Re}\omega$ is the characteristic frequency of an excited wave and $|\mathrm{Im}\omega|=|\mathrm{Im}\delta|$ is its rate of growth.

The expression (1.4) describes circularly polarized waves. The angular frequency $\tilde{\omega}$ and the wave number k may be positive or negative. If $\tilde{\omega}$ is positive, the wave rotates clockwise when the observer is looking in the positive direction (i.e., in the direction of the magnetic field $\mathbf{B}_0$). The phase velocity of the wave is $\tilde{\omega}/k$. The sign of $\tilde{\omega}/k$ indicates the direction of propagation of the wave, i.e., if $\tilde{\omega}/k>0$, the wave moves in the direction of the magnetic field, and if $\tilde{\omega}/k<0$, the wave moves in the direction opposite to that of the magnetic field.

A circularly polarized wave has a positive or a negative helicity. The term positive helicity designates a wave in which the electric vector rotates clockwise as the wave moves away from the observer. For such a wave designated as an H_+ wave, one has $\tilde{\omega}>0$ and $\tilde{\omega}/k>0$ or $\tilde{\omega}<0$ and $\tilde{\omega}/k<0$. On the other hand, for an H_- wave having negative helicity, one has $\tilde{\omega}>0$ and $\tilde{\omega}/k<0$ or $\tilde{\omega}<0$ and $\tilde{\omega}/k>0$. Hence, for a fixed sign of $\tilde{\omega}$ the sign of the wave number k determines the helicity of the wave.[7] For $k>0$, one has a wave of positive helicity or an H_+ wave, and for $k<0$, one has a wave of negative helcity or an H_- wave. The wave having positive helcity is often designated as "left-handed polarized wave."[15]

3. Resonant Waves

Consider the term σF_b which represents the contribution of the beam to the dispersion equation (1.4). It has been assumed that the intensity of the beam is small ($\sigma\ll 1$). Therefore, if the expression F_b is bounded, then for sufficiently small values of σ the contribution of the term σF_b to the dispersion equation (1.4) is negligible. Thus the effect of the beam is significant only in the neighborhood of singularities of the expression F_b. Motivated by this consideration, waves will be analyzed for which the characteristic frequency $\mathrm{Re}\omega$ satisfies the condition

$$\mathrm{Re}\omega\approx\tilde{\omega}=ck\beta_{\shortparallel}+\Omega_e(1-\beta^2)^{1/2}. \tag{1.11}$$

This expression represents a resonance between the characteristic frequency of the excited wave and the Doppler shifted electron gyrofrequency of the electrons in the beam.

A resonance condition similar to (1.11) for a plasma-beam system in which the electrons move in the direction parallel to that of $\mathbf{B}_0$ (i.e., when $\beta_\perp=0$) was formulated, apparently for the first time, in an unpublished report by Dawson and Bernstein.[16]

B. Analysis of the Dispersion Equation

1. P and B Waves

In seeking to understand more fully the mechanism of the excitation of waves by a helical electron beam, some insight can be gained by comparing the "particle problem" with the "beam problem." The particle problem deals with the radiation produced by a single particle moving along a helical trajectory around a line of the magnetic field. The beam problem is concerned with the radiation produced by a beam, i.e., it takes into account the collective effects due to an assembly of charged particles gyrating in the magnetic field. In the particle problem there are no stability considerations and the energy flow is represented by a nonzero Poynting vector which is real. There are no complex quantities that would indicate growth or decay. On the other hand, the beam problem is based on stability considerations and is expressed by the emergence of a growing excited wave which results from an instability.

It will be shown that as a result of the perturbation produced by the beam there are two types of growing waves which occur in a plasma-beam system: a P wave which can be directly associated with a wave produced by a single particle and remaining waves designated as B waves.

Let ω', k' represent the frequency and the wave number of a wave which is radiated in the wave zone by a single helical particle and which moves in the direction of the magnetic field. The frequency of such a wave will be equal to the Doppler shifted angular frequency of the particle. Furthermore, ω' and k' have to satisfy the dispersion equation for the unperturbed plasma.

Consequently,

$$\omega'=\tilde{\omega};\quad F_p(\omega',k')=0. \tag{1.12}$$

Consider now the collective effect of an assembly of helical electrons which form a beam. Let ω'' be the frequency of a wave in a plasma-beam system with a wave

[13] E. S. Weibel, Phys. Rev. Letters **2**, 83 (1959).
[14] See, for instance, A. I. Akhiezer and IA. B. Fainberg, Zh. Eksperim. i Teor. Fiz. **21**, 1262 (1951).
[15] See, for instance, J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), p. 206.
[16] J. Dawson and I. B. Bernstein, paper presented at the Controlled Thermonuclear Conference, D. C., TID-7558, 360, 1958 (unpublished); see also I. B. Bernstein and K. Trehan, Nucl. Fusion **1**, 3 (1960).

number k', i.e., the same wave number as in the case of the particle problem. The quantities ω'', k' must satisfy the dispersion equation for the plasma-beam system, i.e.,

$$F_p(\omega'',k')-\sigma F_b(\omega'',k')=0. \tag{1.13}$$

One has in accordance with (1.9)

$$\omega''=\tilde{\omega}+\delta \tag{1.14}$$

and in accordance with (1.10)

$$\lim_{\sigma\to 0}\omega''=\tilde{\omega}. \tag{1.15}$$

It will be shown that there are two sets of solutions ω''. One set satisfies the relationship

$$\lim_{\sigma\to 0}\omega''=\omega', \tag{1.16}$$

and the other set satisfies the relationship

$$\lim_{\sigma\to 0}\omega''\neq\omega'. \tag{1.17}$$

Thus, for $\sigma\to 0$, the set defined by the relationship (1.16) represents excited waves having characteristic frequencies which differ very little from the corresponding stationary waves radiated by a single particle in the unperturbed plasma. These are the P waves which satisfy the relationship

$$(F_p)_{\omega=\tilde{\omega}}=0. \tag{1.18}$$

On the other hand, the relationship (1.17) defines waves which do not approach in the limit any waves which can be radiated by a single particle. These latter waves are the B waves and they do not satisfy the relationship (1.18).

An alternate definition of a P wave excited by a beam with velocity components $\beta_{\parallel}$ and $\beta_{\perp}$ is as follows: a P wave is a wave with a wave number k which is equal to that of a wave radiated in the unperturbed plasma by a single particle having velocity components $\beta_{\parallel}$ and $\beta_{\perp}$.

2. *Analysis of the Dispersion Equation*

Considering (1.9) and (1.11), Eq. (1.4) can be expressed in the form

$$(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}\delta^3+(F_p)_{\omega=\omega}\delta^2 \\ -\tfrac{1}{2}[\sigma\omega_e^2(1-\beta^2)^{1/2}\beta_{\perp}^2(c^2k^2-\tilde{\omega}^2)]=0 \tag{1.19}$$

subject to the three conditions

$$|\delta(2\tilde{\omega}+\delta)|\ll|c^2k^2-\tilde{\omega}^2|, \tag{1.20}$$

$$|\delta(\Omega_e(1-\beta^2)^{1/2}+\delta)|\ll|\beta_{\perp}^2(c^2k^2-\tilde{\omega}^2)/2|, \tag{1.21}$$

and

$$F_p(\omega)=F_p(\tilde{\omega}+\delta)=(F_p)_{\omega=\tilde{\omega}}+(\partial F_p/\partial\omega)_{\omega=\omega}\delta. \tag{1.22}$$

The condition (1.22) requires that F_p can be approximated by two terms of a Taylor series expansion about $\omega=\tilde{\omega}$.

The inequalities (1.20) and (1.21) impose certain restrictions on the solution of (1.19). Thus, according to (1.21), $\beta_{\perp}$ cannot be arbitrarily small and, according to (1.20), the phase velocity of the excited wave cannot be too close to the velocity of light in vacuum. Conditions on the size of $|\delta|$ to make the expression (1.22) a reasonable approximation can be determined from the expansion of F_p.

Equation (1.19) describes the behavior of the plasma-beam system under the stated assumptions. If there exists a complex value of δ which satisfies simultaneously Eq. (1.19) and the stated assumptions, an instability is indicated.

The coefficients in the equation (1.19) depend on the characteristic frequency $\tilde{\omega}$. Consideration of the discriminant of the cubic indicates that the condition for complex roots of (1.19) is

$$\frac{\sigma\omega_e^2\beta_{\perp}^2(1-\beta^2)^{1/2}[(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}]^2}{8}>\frac{[(F_p)_{\omega=\tilde{\omega}}]^3}{27(c^2k^2-\tilde{\omega}^2)}. \tag{1.23}$$

Thus, for sufficiently small values of σ it is seen that complex roots of (1.19), and hence instability, can occur only at frequencies $\tilde{\omega}$ which satisfy the condition

$$(F_p)_{\omega=\tilde{\omega}}\approx 0, \tag{1.24}$$

or at frequencies which satisfy the condition

$$[(F_p)_{\omega=\tilde{\omega}}](c^2k^2-\tilde{\omega}^2)<0. \tag{1.25}$$

The condition (1.24) is identical to (1.18) discussed above. Therefore, this condition is satisfied if the instability is represented by P waves. Explicit expressions will be derived for two frequency ranges. One of these ranges consists of frequencies $\tilde{\omega}$ near the frequency of a P wave and is determined by

$$|(F_p)_{\omega=\tilde{\omega}}|\ll|(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}\cdot\delta|. \tag{1.26}$$

The other range consists of frequencies $\tilde{\omega}$ such that

$$|(F_p)_{\omega=\tilde{\omega}}|\gg|(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}\cdot\delta| \tag{1.27}$$

and (1.25) both hold. These waves are B-waves.

3. *P Instability*

(*a*) *Rate of growth*: In a P instability the condition (1.26) holds and thus the term containing δ^2 can be neglected in the cubic equation (1.19). Consequently, the following expression is obtained for $|\mathrm{Im}\delta|$:

$$|\mathrm{Im}\delta|=\frac{\sqrt{3}}{2}\sigma^{1/3}\left|\frac{\omega_e^2(1-\beta^2)^{1/2}\beta_{\perp}^2(c^2k^2-\tilde{\omega}^2)}{2(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}}\right|^{1/3}. \tag{1.28}$$

The above expression represents the rate of growth, $|\mathrm{Im}\delta|$, as a function of the frequency $\tilde{\omega}$ and of the wave number k of the excited P wave. Since the wave is in resonance with the beam, the wave number k is related to $\tilde{\omega}$ by the equality

$$k=\tilde{k}=[\tilde{\omega}-\Omega_e(1-\beta^2)^{1/2}]/c\beta_{\parallel} \tag{1.29}$$

obtained from (1.11). Substituting (1.29) in (1.28), the rate of growth can be expressed as follows:

$$|\mathrm{Im}\delta| = \frac{\sqrt{3}}{2}\sigma^{1/3}\left|\frac{\omega_e^2(1-\beta^2)^{1/2}\beta_\perp^2\varphi}{2\psi}\right|^{1/3}, \tag{1.30}$$

where φ is obtained by substituting $k=\tilde{k}$ as formulated in (1.29) into the expression $c^2k^2-\tilde{\omega}^2$ as follows:

$$\varphi \equiv \varphi(\beta_\perp,\beta_{\shortparallel},\tilde{\omega}) = (1/\beta_{\shortparallel}^2)[\tilde{\omega}^2(1-\beta_{\shortparallel}^2) - 2\tilde{\omega}\Omega_e(1-\beta^2)^{1/2}+\Omega_e^2(1-\beta^2)] \tag{1.31}$$

and

$$\psi \equiv \psi(\beta_\perp,\beta_{\shortparallel},\tilde{\omega}) = \left(\frac{\partial F_p}{\partial \omega}\right)_{\omega=\tilde{\omega}} = 2\tilde{\omega} - \frac{\omega_i^2\Omega_i}{(\tilde{\omega}+\Omega_i)^2} + \frac{\omega_e^2\Omega_e}{(\tilde{\omega}-\Omega_e)^2}. \tag{1.32}$$

(*b*) *Frequencies of excited waves*: The frequencies of the excited waves are represented by the roots of the equation $(F_p)_{\omega=\tilde{\omega}}=0$. Taking into account that $k=\tilde{k}$, one can express this situation as

$$\Phi = 0, \tag{1.33}$$

where

$$\Phi \equiv \Phi(\beta_\perp,\beta_{\shortparallel},\tilde{\omega}) = (F_p)_{\omega=\tilde{\omega}} = \tilde{\omega}^2 - \frac{1}{\beta_{\shortparallel}^2}[\tilde{\omega}-\Omega_e(1-\beta^2)^{1/2}]^2 - \frac{\tilde{\omega}^2(\omega_e^2+\omega_i^2)}{(\tilde{\omega}+\Omega_i)(\tilde{\omega}-\Omega_e)}. \tag{1.34}$$

4. *B Instability*

Since, in a B instability, the inequality (1.27) holds, the term containing δ^3 in the cubic equation (1.19) can be neglected and the following expression is obtained:

$$\delta^2 = \tfrac{1}{2}\sigma\omega_e^2(1-\beta^2)^{1/2}\beta_\perp^2(\varphi/\Phi). \tag{1.35}$$

In order to have an instability, δ^2 should be negative. Consequently, it is necessary that

$$\varphi/\Phi < 0. \tag{1.36}$$

In such case the rate of growth of the excited wave is as follows:

$$|\mathrm{Im}\delta| = \frac{\sqrt{2}}{2}\sigma^{1/2}\left|\frac{\omega_e^2(1-\beta^2)^{1/2}\beta_\perp^2\varphi}{\Phi}\right|^{1/2}. \tag{1.37}$$

Thus, if $\beta_\perp$ and $\beta_{\shortparallel}$ are known, one has an instability for frequencies ω satisfying the inequality (1.36) and the rates of growth corresponding to these functions are given by the expression (1.37).

5. *Rate of Growth in "P" and "B" Instabilities*

In accordance with (1.28), the rate of growth of a "P" wave can be expressed as

$$|\mathrm{Im}\delta| = \sigma^{1/3}K_1, \tag{1.38}$$

where K_1 is a finite valued function of $\beta_\perp$, $\beta_{\shortparallel}$, ω_i, ω_e, Ω_i, and Ω_e. This function is independent of σ. Hence, for $\sigma \to 0$, $|\mathrm{Im}\delta|$ tends to zero as the $\frac{1}{3}$ power of σ.

Using the relationship (1.37) the rate of growth of a "B" wave can be expressed as

$$|\mathrm{Im}\delta| = \sigma^{1/2}K_2, \tag{1.39}$$

where K_2 is a finite valued function of $\beta_\perp$, $\beta_{\shortparallel}$, ω_i, Ω_i, and Ω_e. This function is independent of σ. Hence, for $\sigma \to 0$, $|\mathrm{Im}\delta|$ tends to zero as the $\frac{1}{2}$ power of σ.

Clearly for sufficiently small values of σ, the rate of growth expressed by (1.38) is large when compared with the rate of growth expressed by (1.39). Consequently, in previous investigations the P instability has been designated as "strong" and the B instability has been designated as "weak."[8] In practical applications, however, σ cannot be arbitrarily small due to the physical nature of the system. Consequently, in certain cases the designations of instabilities as "strong" and "weak" on the basis of the smallness of σ may not be justified. Thus, for a fixed value of σ it may be possible to have a combination of the other parameters which satisfy all necessary conditions but are such that the growth rate of "P" waves may not be considerably larger than the growth rate of a "B" wave. That is, even though $\sigma^{1/2}$ may be much less than $\sigma^{1/3}$ the parameters of the system may cause the coefficient of $\sigma^{1/2}$ to be enough larger than the coefficient of $\sigma^{1/3}$ to offset partly this difference.

6. *Dispersion Equation in the Limit of $\sigma \to 0$*

Consideration of circumstances arising as the beam intensity σ approaches zero has been a motivating factor in the above. Hence it is of interest to determine the limiting form of the dispersion equation of the plasma-beam system as σ tends to zero which can be expressed by

$$\lim_{\sigma\to 0}(F_p - \sigma F_b).$$

Consider the expression for F_b as formulated in (1.6). Because of the inequality (1.21) the first term in the brackets of (1.6) can be neglected when compared to the second term and consequently F_b has the form

$$F_b = M/\delta^2,$$

where M is independent of δ.

Suppose there is a P instability. It is seen from (1.38)

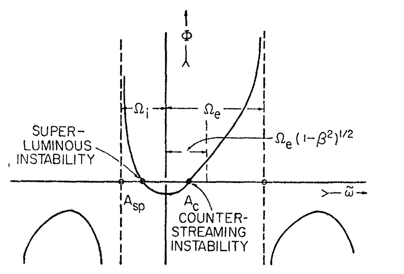


FIG. 1. Graph of $\Phi(\beta_\perp,\beta_{\shortparallel},\tilde{\omega})$. Double P instability.

that δ is of the order of the $\frac{1}{3}$ power of σ. Thus, the quantity σF_b is of the order of $\sigma^{1/3}$ and hence tends to zero with σ. Therefore, the dispersion equation (1.4) for the plasma-beam system degenerates into the dispersion equation (1.8) for the stationary plasma as $\sigma \to 0$. This can be expressed as

$$\lim_{\sigma\to 0}(F_p - \sigma F_b) = F_p. \tag{1.40}$$

Suppose there is a B instability. It is seen from (1.39) that δ is of the order of $\sigma^{1/2}$. Therefore, the expression σF_b does not tend to zero with σ. In such case the dispersion equation (1.4) for the plasma-beam system does not degenerate into the dispersion equation for the stationary plasma as $\sigma \to 0$. This can be expressed as

$$\lim_{\sigma\to 0}(F_p - \sigma F_b) \neq F_p. \tag{1.41}$$

The expressions (1.40) and (1.41) display the fundamental difference in the character of B and P instabilities.

II. MULTIPLE EXCITATION IN A *P* INSTABILITY

A. Occurrence of a Multiple Instability

Consider the equation $\Phi = 0$ as formulated in (1.33) and (1.34). For any fixed values of $\beta_\perp$ and $\beta_{\parallel}$ this equation yields at least two real roots in $\tilde{\omega}$, and, therefore, there are at least two waves having different frequencies which are excited by the beam. Consequently, there is a multiple instability.

The character of the multiple instability can be determined from the behavior of the function

$$\Phi \equiv \Phi(\beta_\perp, \beta_{\parallel}, \tilde{\omega}) \tag{2.1}$$

defined in (1.34) and illustrated qualitatively by the curves in Figs. 1, 2, and 3(a). The graphs shown in these figures correspond, respectively, to three different sets of values of $\beta_\perp$ and $\beta_{\parallel}$. The frequencies of the excited P waves correspond to the roots of $\Phi = 0$ and are, therefore, determined by the points of intersection of each graph in Figs. 1, 2, and 3(a) with the axis of abscissas (the $\tilde{\omega}$ axis). In Fig. 1 the quantities $\beta_{\parallel}$ and $\beta_\perp$ have been chosen so as to provide two points of intersection of the graph with the $\tilde{\omega}$ axis. Therefore, there are two waves having different frequencies which are exicted in the plasma-beam system (two instabilities). The points representing these instabilities are labeled A_{sp} and A_c, respectively. In Fig. 2 or in Fig. 3(a)

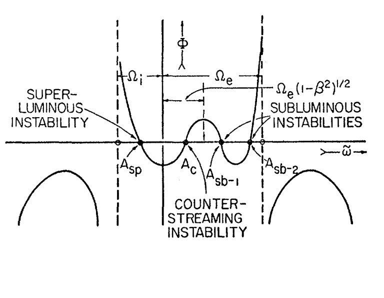


FIG. 2. Graph of $\Phi(\beta_\perp,\beta_{\parallel},\tilde{\omega})$. Quadruple P instability. Subluminous excitation is represented by frequencies below the electron gyrofrequency of the stationary plasma.

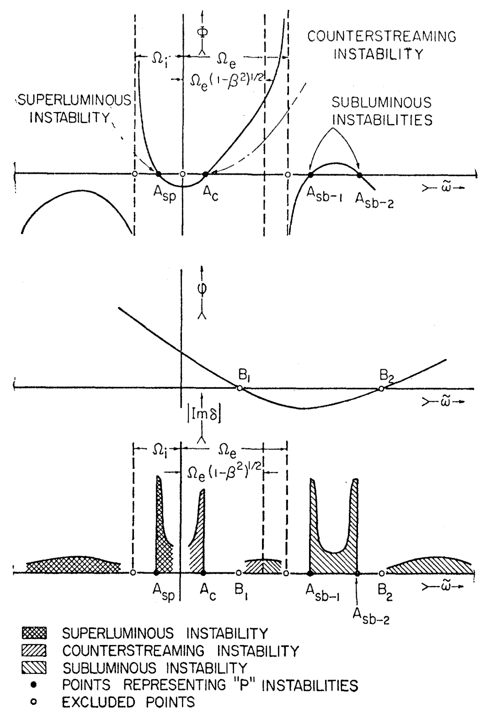


FIG. 3. (a) Graph of $\Phi(\beta_\perp,\beta_{\parallel},\tilde{\omega})$. Quadruple P instability. Subluminous excitation is represented by frequencies above the electron gyrofrequency. (b) Graph of $\varphi(\beta_\perp,\beta_{\parallel},\tilde{\omega})$. The above graph intersects the $\tilde{\omega}$ axis at points B_1 and B_2 which represent frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$, respectively. These frequencies are as follows: $\tilde{\omega}_1 = \Omega_e(1-\beta^2)^{1/2}/(1+\beta_{\parallel})$, and $\tilde{\omega}_2 = \Omega_e(1-\beta^2)^{1/2}/(1-\beta_{\parallel})$. (c) Relationship between $|\mathrm{Im}\,\delta|$ and $\tilde{\omega}$. Quadruple P instability.

there are four points of intersection. Thus, Fig. 2 or Fig. 3(a) shows four instabilities, each of which gives rise to a growing wave having a distinct characteristic frequency (quadruple instability). The points representing these instabilities are labeled A_{sp}, A_c, A_{sb-1}, and A_{sb-2}, respectively. Figure 3(c) will be discussed later in Sec. IV. [The scales in Figs. 1–3(c) are considerably distorted in order to show the qualitative features of the graphs.]

B. Classification of Instabilities

In the limiting case of $\sigma \to 0$ the frequency ω of the excited wave tends to $\tilde{\omega}$. The classification of instabilities discussed in this section is based on the relationship between the velocity $c\beta_{\parallel}$ of the beam and the phase velocity $\tilde{\omega}/k$ of the wave excited by the beam. In accordance with this classification, the instability labeled A_{sp} is "superluminous," the one labeled A_c is "counterstreaming," and the instabilities labeled A_{sb-1} and A_{sb-2} are "subluminous."

1. Superluminous Instability (point A_{sp})

An instability is superluminous if the beam moves in the same direction as the excited wave, and the velocity $c\beta_{\parallel}$ of the beam exceeds the phase velocity $\tilde{\omega}/k$ of the wave. It will be shown that the point A_{sp} in Figs. 1, 2, and 3(a) represents such an instability.

The frequency $\tilde{\omega}$ represented by the point labeled A_{sp} satisfies the following inequalities:

$$\tilde{\omega}<0 \tag{2.2}$$

and

$$|\tilde{\omega}|<\Omega_i. \tag{2.3}$$

Therefore, it follows from the relationship (1.11) and from the inequality (2.2) that

$$k\beta_{\parallel}<0. \tag{2.4}$$

The quantity $\beta_{\parallel}$ in the inequality (2.4) may be positive or negative. Assume that

$$\beta_{\parallel}>0, \tag{2.5}$$

i.e., the helical beam moves in the direction of the magnetic field $\mathbf{B}_0$. Then one has

$$k<0. \tag{2.6}$$

Taking into account (2.2) and (2.6), one obtains

$$\tilde{\omega}/k>0, \tag{2.7}$$

i.e., the phase velocity is positive. Consequently, the excited wave moves in the same direction as the beam.

It can be seen also from the relationship (1.11) and the inequalities (2.2) and (2.4) that

$$c\beta_{\parallel}>\tilde{\omega}/k. \tag{2.8}$$

Consequently, the velocity of the beam exceeds the phase velocity of the wave.

Using a similar argument, it can be shown that the instability represented by the point A_{sp} is also superluminous when the beam moves against the direction of $\mathbf{B}_0(\beta_{\parallel}<0)$.

Consequently, whenever the excited frequency is negative ($\tilde{\omega}<0$), the instability is superluminous for a beam moving in the positive or negative direction. If the direction of the beam is positive ($\beta_{\parallel}>0$), the superluminous instability results in the excitation of a wave having positive helicity (H_+ wave). For a beam moving in the negative direction ($\beta<0$), one has $k<0$ and, therefore, and H_- wave is excited.

2. *Counterstreaming Instability (point A_c)*

The counterstreaming instability is produced by a beam moving in the direction opposite to that of the excited wave. It will be shown that the point A_c in Figs. 1, 2, and 3(a) represents such an instability.

The frequency $\tilde{\omega}$ represented by A_c satisfies the following inequalities:

$$\tilde{\omega}>0 \tag{2.9}$$

and

$$\tilde{\omega}<\Omega_e(1-\beta^2)^{1/2}. \tag{2.10}$$

Taking into account (2.9), (2.10), and (1.11), one obtains

$$k\beta_{\parallel}<0. \tag{2.11}$$

Consequently, if the velocity of the beam is positive ($\beta_{\parallel}>0$), one has $k<0$, and, therefore,

$$\tilde{\omega}/k<0, \tag{2.12}$$

i.e., the wave moves in the direction opposite to that of the beam. It can also be shown that for $\beta_{\parallel}<0$, the direction of the wave is opposite to that of the beam.

For $\beta_{\parallel}>0$ a counterstreaming instability results in the excitation of an H_- wave, whereas for $\beta_{\parallel}<0$, an H_+ wave is excited.

3. *Subluminous Instabilities (points A_{sb-1} and A_{sb-2})*

An instability is subluminous if the beam moves in the same direction as the excited wave, and the velocity of the beam is lower than the phase velocity $\tilde{\omega}/k$ of the wave. It will be shown that the points A_{sb-1} and A_{sb-2} in Figs. 1, 2, and 3(a) represent such an instability.

The frequencies $\tilde{\omega}$ represented by A_{sb-1} and A_{sb-2} satisfy the inequality

$$\tilde{\omega}>\Omega_e(1-\beta^2)^{1/2}. \tag{2.13}$$

It follows from the inequality (2.13) and from the relationship (1.11) that

$$k\beta_{\parallel}>0. \tag{2.14}$$

The quantity $\beta_{\parallel}$ in the inequality (2.14) may be positive or negative. Assume that

$$\beta_{\parallel}>0. \tag{2.15}$$

One has then

$$k>0 \tag{2.16}$$

and, consequently, one obtains from $\tilde{\omega}>0$ and the inequality (2.16) that

$$\tilde{\omega}/k>0, \tag{2.17}$$

i.e., the phase velocity is positive and, therefore, the wave moves in the same direction as the beam.

Using the relationship (1.11) and the inequality (2.13), one obtains

$$c\beta_{\parallel}<\tilde{\omega}/k. \tag{2.18}$$

Consequently, the velocity of the beam is lower than that of the wave.

It can be shown by means of a similar argument that the instability represented by the points A_{sb-1} and A_{sb-2} is also subluminous when the beam moves against the direction of $\mathbf{B}_0$ ($\beta<0$).

For $\beta_{\parallel}>0$ a subluminous instability results in the excitation of an H_- wave, whereas for $\beta_{\parallel}<0$, an H_+ wave is excited.

A superluminous instability gives rise to a wave rotating in the same direction as the perturbed stationary ions (i.e., $\tilde{\omega}<0$), whereas a counterstreaming or a subluminous instability gives rise to a wave rotating in the same direction as perturbed stationary electrons (i.e., $\omega > 0$).

III. FREQUENCIES AND RATES OF GROWTH IN A "*P*" (STRONG) INSTABILITY

A. General

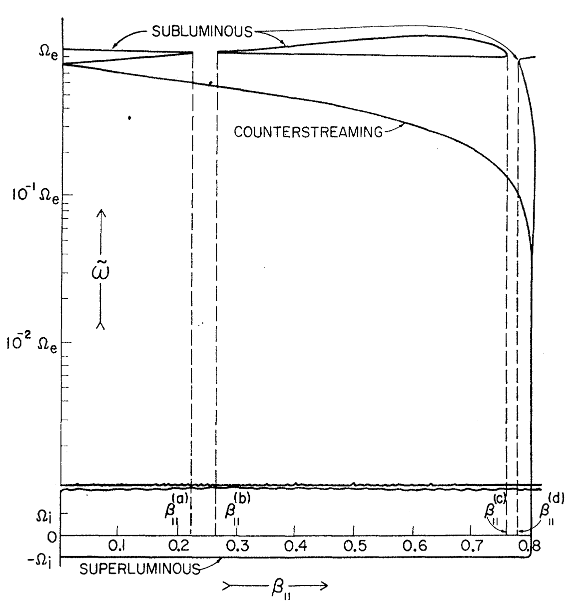
Consider now various types of stationary plasmas which may interact with a helical electron beam. A very useful classification of plasmas based on two nondimensional parameters

$$A^2 = \omega_i^2 / \Omega_i^2 \tag{3.1}$$

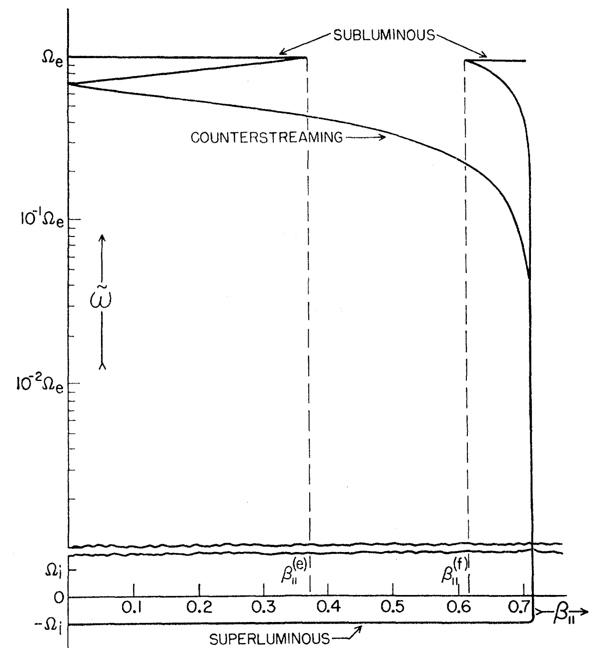
and

$$\eta = M_i / m \tag{3.2}$$

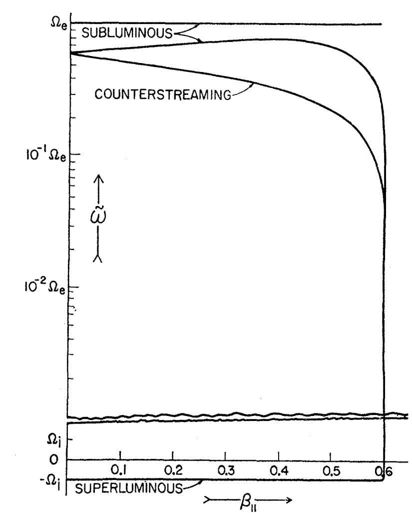
was introduced by Denisse and Delcroix.[5] According to this classification, a plasma is "very rarefied" when $A^2 < \eta^{-1}$; "rarefied" when $\eta^{-1} < A^2 < 1$; is of "small density" when $1 < A^2 < \eta$; and is "dense" when $A^2 > \eta$. Very rarefied and rarefied plasmas are not of particular practical significance at the present time. Such plasmas are usually found only in evacuated vessels in the presence of a very strong magnetic field such as in cyclotrons, etc. Most of the plasmas which occur in nature or which are produced in a laboratory are of small density or dense.



(a)



(b)



(c)

FIG. 4. (a) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_{\perp}=0.6$. (b) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_{\perp}=0.7$. (c) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_{\perp}=0.8$.

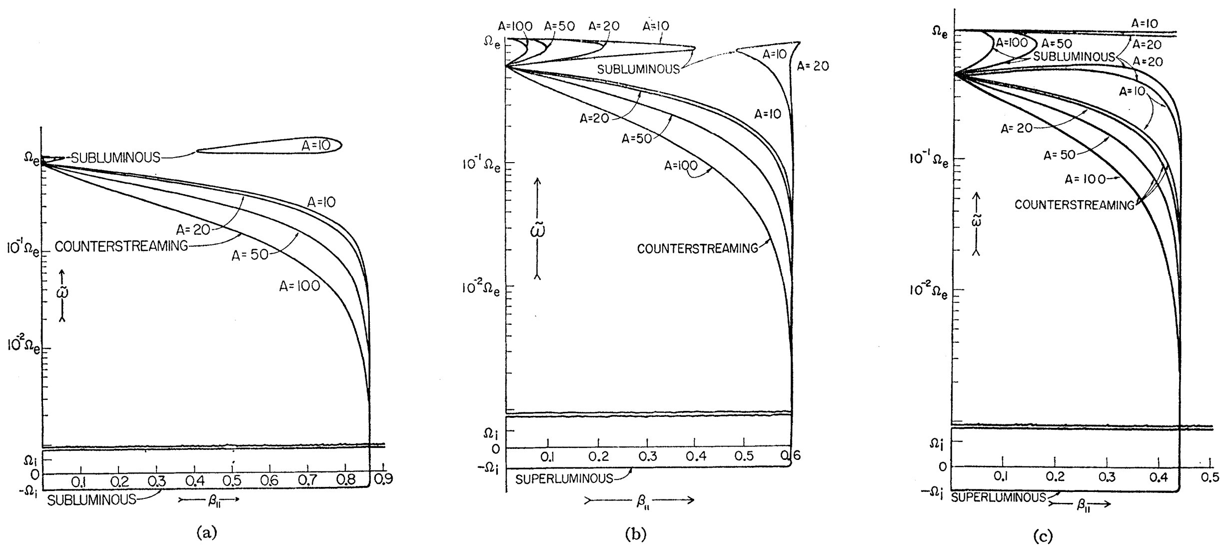


FIG. 5. (a) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.5$, and $A=10$, 20, 50, and 100. (b) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.8$, and $A=10$, 20, 50, and 100. (c) Relationship between the frequency $\tilde{\omega}$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.9$, and $A=10$, 20, 50, and 100.

The interaction of a stationary plasma with a helical electron beam will now be described in terms of the parameters A and η characterizing the plasma and the parameters $\beta_{\perp}$ and $\beta_{\parallel}$ characterizing the beam.

B. Frequencies of Excited Waves

1. Graphical Representation of the Relationship Between the Frequency $\tilde{\omega}$ of an Excited Wave and the Velocity $\beta_{\parallel}$ of the Beam

Using a nondimensional quantity,

$$X=\tilde{\omega}/\Omega_e, \qquad (3.3)$$

representing the frequency of an excited wave and the parameters A, η, $\beta_{\perp}$, and $\beta_{\parallel}$, characterizing a plasma-beam system, the following relationship is obtained from (1.33):

$$X^2-\frac{1}{\beta_{\parallel}^2}[X-(1-\beta^2)^{1/2}]^2-\frac{A^2X^2(1+\eta)}{\eta(\eta X+1)(X-1)}=0. \qquad (3.4)$$

If A, η, and $\beta_{\perp}$ are fixed, one can obtain from (3.4) the relationship between the excited frequency X (or $\tilde{\omega}$) and the velocity $\beta_{\parallel}$ of the beam. Such a relationship is represented graphically in Figs. 4(a), 4(b), and 4(c) in which the excited frequencies are plotted as ordinates and the beam velocities are plotted as abscissas. The three types of instability produced by the beam are clearly illustrated by means of three curves labeled "superluminous," "counterstreaming," and "subluminous."

In all three systems illustrated in Figs. 4(a), 4(b), and 4(c), the stationary plasma is assumed to be the same and is defined by $A=1$ and $\eta=1837$. (These values of A and η are particularly convenient in order to represent within the scale of the drawings the characteristic features of the plasma-beam interaction.) It should be noted that the helical electron beam interacting with the plasma is different in each of the cases illustrated in these figures. In Fig. 4(a) $\beta_{\perp}=0.6$, and the corresponding values of $\beta_{\perp}$ in Figs. 4(b) and 4(c) are $\beta_{\perp}=0.7$ and $\beta_{\perp}=0.8$, respectively. The abscissas in each of these graphs cover progressively decreasing range which extends from $\beta_{\parallel}=0$ to $\beta_{\parallel}=(1-\beta_{\perp}^2)^{1/2}$.

2. Three Types of Plasma-Beam Systems

The comparison of Figs. 4(a), 4(b), and 4(c) shows the general trend in the behavior of the plasma-beam interaction for increasing values of $\beta_{\perp}$. In order to facilitate further discussion on this subject, the plasma-beam system illustrated in Fig. 4(a) will be referred to as belonging to type I, whereas the systems illustrated in Figs. 4(b) and 4(c) will be referred to as belonging to types II and III, respectively. The characteristic features differentiating the above plasma-beam systems are as follows:

(a) In a system of type III the instability is always quadruple, i.e., there are always four waves excited by a beam having any fixed velocity $\beta_{\parallel}$. On the other hand, in a system of type I or type II there are certain velocity ranges in which the instability is quadruple and other velocity ranges in which the instability is double.

(b) In a system of type II the subluminous instabilities are represented by frequencies which are always below the electron gyrofrequency of the stationary plasma. Therefore, these frequencies satisfy the inequality

$$\Omega_e(1-\beta^2)^{1/2}<\tilde{\omega}<\Omega_e. \tag{3.5}$$

(c) In a system of type I there are two frequency ranges which characterize a subluminous instability. One of these is represented by (3.5), and the other is represented by the inequality

$$\tilde{\omega}>\Omega_e. \tag{3.6}$$

Thus, in a plasma-beam system of type I a subluminous instability may be represented by a frequency which exceeds the electron gyrofrequency of the stationary plasma.

In a different graphical representation discussed in a previous section, the subluminous instabilities characterized by frequencies in the range (3.5) are shown in Fig. 2, whereas the subluminous instabilities characterized by frequencies in the range (3.6) are shown in Fig. 3(a).

It is noted that whenever the instability is double, the two excited waves represent, respectively, a superluminous and counterstreaming instability. On the other hand, in case of a quadruple instability one excited wave represents a superluminous instability, another excited wave represents a counterstreaming instability, and there are two excited waves representing a subluminous instability.

Plasma-beam systems of type I are characterized by two velocity regions in which there is a double instability. These regions are represented, respectively, in Fig. 4(a) as follows:

$$\beta_{\parallel}{}^{(a)}<\beta_{\parallel}<\beta_{\parallel}{}^{(b)} \tag{3.7}$$

and

$$\beta_{\parallel}{}^{(c)}<\beta_{\parallel}<\beta_{\parallel}{}^{(d)}. \tag{3.8}$$

On the other hand, in plasma-beam systems of type II there is only one velocity region in which the instability is double. This region is represented in Fig. 4(b) as follows:

$$\beta_{\parallel}{}^{(e)}<\beta_{\parallel}<\beta_{\parallel}{}^{(f)}. \tag{3.9}$$

3. "Critical" Values of the Transverse Velocity Component $\beta_\perp$

A numerical analysis has been performed on various plasma-beam systems characterized by a wide range of values of A and η which, according to the classification of Denisse and Delcroix,[5] represent the majority of all typical plasmas encountered in nature or produced in a laboratory. One has $\eta=1837$ for a plasma comprised of hydrogen ions (such as solar corona and interstellar clouds), $\eta=7348$ for helium ions, and $\eta\sim 58\,784$ for molecular oxygen ions encountered in the ionosphere and in electrical discharges through air. It can be concluded from this analysis that for a given value of A and η there are two critical values $\beta_\perp'$ and $\beta_\perp''$ of $\beta_\perp$ where $\beta_\perp'<\beta_\perp''$ and such that plasma-beam systems of types I, II, and III are characterized by $\beta_\perp>\beta_\perp''$, $\beta_\perp''>\beta_\perp>\beta_\perp'$, and $\beta_\perp<\beta_\perp'$, respectively.

The relationship between the beam velocity $\beta_{\parallel}$ and the frequency $\tilde{\omega}$ of an excited wave in other typical plasma-beam systems is represented graphically in Figs. 5(a), 5(b), and 5(c).

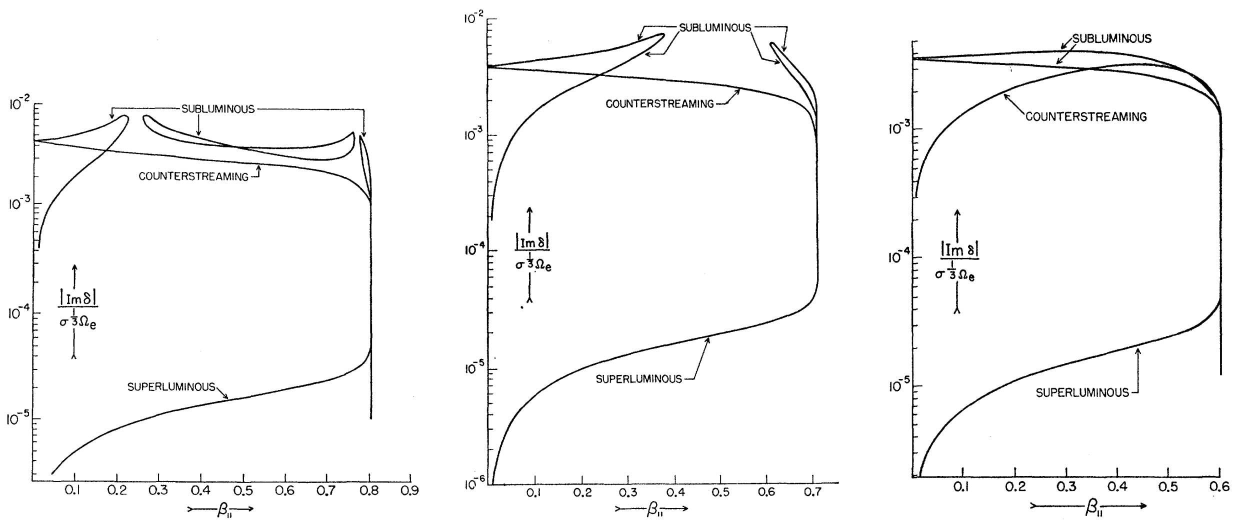


FIG. 6. (a) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_\perp=0.6$. (b) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_\perp=0.7$. (c) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in a plasma-beam system characterized by $A=1$, $\eta=1837$, and $\beta_\perp=0.8$.

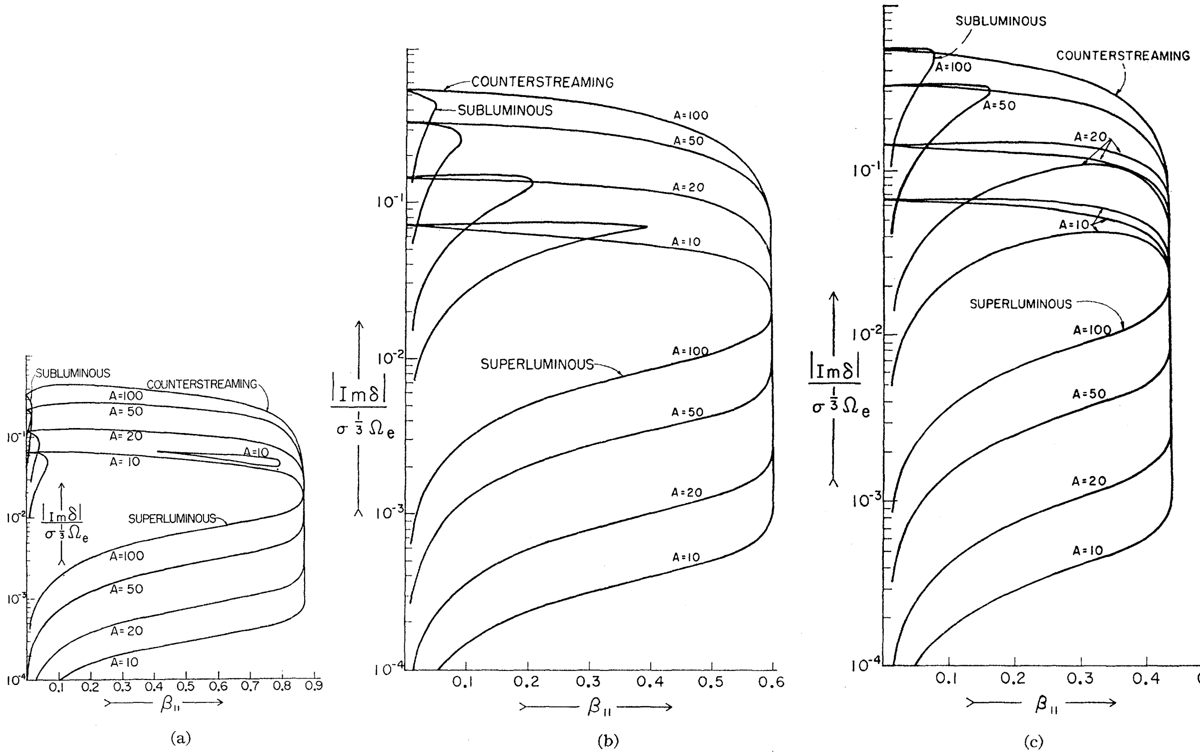


FIG. 7. (a) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.5$, and $A=10$, 20, 50, and 100. (b) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.8$, and $A=10$, 20, 50, and 100. (c) Relationship between the rate of growth $Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e$ of an excited P wave and the corresponding beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.9$, and $A=10$, 20, 50, and 100.

C. Rate of Growth of Excited Waves

Using a nondimensional quantity,

$$Y=|\mathrm{Im}\delta|/\sigma^{1/3}\Omega_e \tag{3.10}$$

representing the rate of growth of an excited wave, the following equality is obtained from (1.30):

$$Y=\frac{\sqrt{3}}{2}\left|\frac{A^2(1-\beta^2)^{1/2}\beta_{\perp}^2[(X-(1-\beta^2)^{1/2})^2-X^2\beta_{\parallel}^2]}{2\eta\beta_{\parallel}^2[2X-(A^2/\eta(\eta X+1)^2)+(A^2/\eta(X-1)^2)]}\right|^{1/3}. \tag{3.11}$$

The above equality represents a relationship between the quantities Y and X characterizing an excited wave and the parameters A, η, $\beta_{\perp}$, and $\beta_{\parallel}$ representing the plasma-beam system. Combining (3.4) and (3.11), one can eliminate X and obtain a relationship expressing Y as a function of the plasma parameters A,η and the beam parameters $\beta_{\perp}$, $\beta_{\parallel}$.

If A, η, and $\beta_{\perp}$ are fixed, one can obtain the relationship between the rate of growth Y (or $|\mathrm{Im}\delta|$ if one assumes that σ and Ω_e are fixed) and the velocity $\beta_{\parallel}$ of the beam. Such a relationship is represented graphically in Figs. 6(a), 6(b), and 6(c). The quantities $A=1$, $\eta=1837$, and $\beta_{\perp}=0.6$ are the same in Figs. 4(a) and 6(a). Similarly, one has $A=1$, $\eta=1837$, and $\beta_{\perp}=0.7$ in Figs. 4(b) and 6(b), and $A=1$, $\eta=1837$, and $\beta_{\perp}=0.8$ in Figs. 4(c) and 6(c).

The relationship between the beam velocity $\beta_{\parallel}$ and the rate of growth Y (or $|\mathrm{Im}\delta|$) of the excited waves for the values of A and η in Figs. 5(a), 5(b), and 5(c) is represented graphically in Figs. 7(a), 7(b), and 7(c), respectively.

It is seen from the above graphs that the rate of growth characterizing a superluminous instability is always smaller than the corresponding rate of growth in a subluminous or a counterstreaming instability.

D. Index of Refraction and Velocity Index

Consider now the phase velocities $\tilde{\omega}/k$ of the characteristic waves excited by a helical beam. The properties of a plasma-beam system may be described by means of a velocity index

$$S=ck\beta_{\parallel}/\tilde{\omega}, \tag{3.12}$$

representing the ratio of the beam velocity of $c\beta_{\parallel}$ to the phase velocity of the characteristic wave, or by means of a refractive index

$$N=|ck/\tilde{\omega}| \tag{3.13}$$

representing the magnitude of the ratio of the velocity of light in a vacuum to the phase velocity of the characteristic wave.

Figure 8 illustrates graphically the relationship between the velocity $\beta_{\parallel}$ of the helical beam and the absolute value of the index S characterizing various waves excited by the beam. It is apparent that in a superluminous instability one has $S>1$, whereas in a subluminous and counterstreaming instability one has $0<S<1$ and $S<0$, respectively. Figure 9 illustrates graphically the relationship between the velocity $\beta_{\parallel}$ and the index N characterizing various waves excited by the beam. Both Figs. 8 and 9 apply to plasma-beam systems characterized by $\eta=1837$, $\beta_{\perp}=0.8$, and $A=10$, 20, 50, and 100.

IV. COMPARISON OF "P" (STRONG) AND "B" (WEAK) INSTABILITIES

It will be shown that both P and B instabilities occur simultaneously in a system excited by a helical beam. Some of the B instabilities occupy frequency ranges in the immediate neighborhood of frequencies which char-

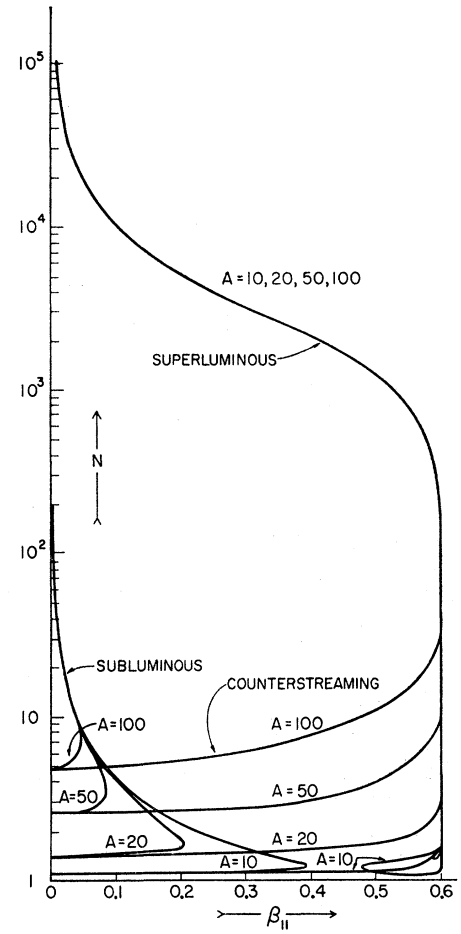


FIG. 9. Relationship between the refractive index $N=|ck/\omega|$ and the beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $A=10$, 20, 50, and 100, $\eta=1837$, and $\beta_{\perp}=0.8$.

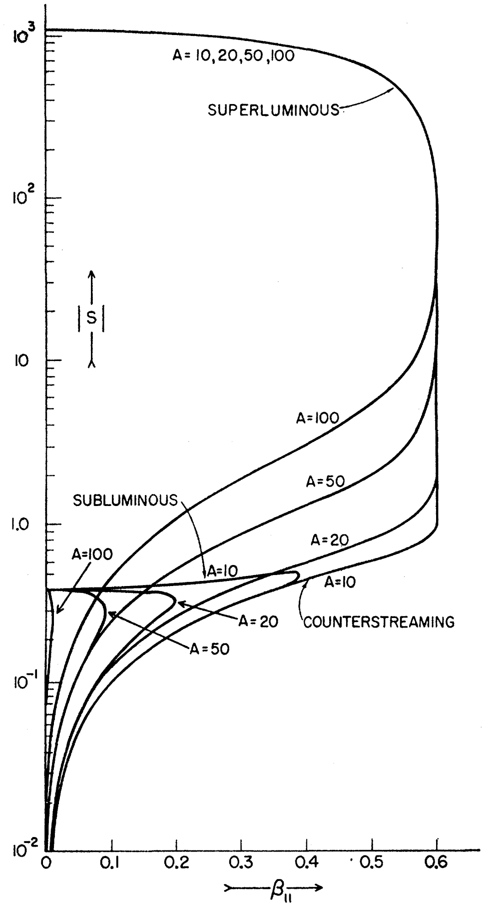


FIG. 8. Relationship between the absolute value of the velocity index $S=ck\beta_{\parallel}/\omega$ and the beam velocity $\beta_{\parallel}$ in plasma-beam systems characterized by $A=10$, 20, 50, and 100, $\eta=1837$, and $\beta_{\perp}=0.8$.

acterize the corresponding P instability. Therefore, they represent "continuations" of the P instabilities along the frequency axis. These continuations are extended either into the region of higher frequencies or into the region of lower frequencies. A similar continuation of a P instability into the region of B instabilities has been described in previous investigations.[8] In these previous investigations the P instability resulting from a "strong interaction" was designated as "strong." The B instability resulting from a "weak interaction" was designated as "weak."

Consider Eq. (1.19). If various parameters defining the plasma-beam system are fixed, this equation yields an expression of the form

$$|\mathrm{Im}\delta|=\zeta(\tilde{\omega}), \tag{4.1}$$

in which the rate of growth is represented as an explicit function of the frequency of the excited wave. A qualitative behavior of the function (4.1) is illustrated graphically in Fig. 3(c).

The general behavior of the graph in Fig. 3(c) can be readily ascertained from the inspection of Figs. 3(a)

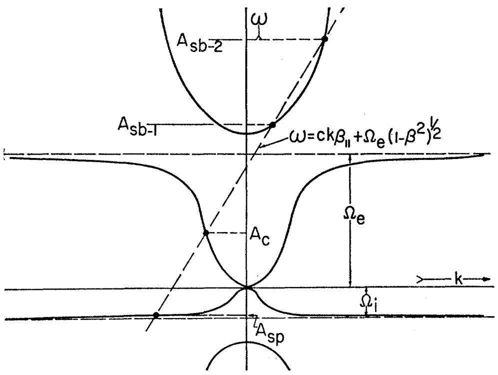


FIG. 10. "$\omega - k$" diagrams for an unperturbed plasma and for the resonant beam.

and 3(b). It should be recalled here that the graph of Fig. 3(a) represents the function Φ formulated in (1.34). The graph of Fig. 3(b) represents the function φ formulated in (1.31).

The P instabilities can be readily identified in Fig. 3(c) since they correspond to the points A_{sp}, A_c, A_{sb-1}, and A_{sb-2} at which the graph of Fig. 3(b) intersects the $\tilde{\omega}$ axis.

The B instabilities shown in Fig. 3(c) can be ascertained from the inspection of Figs. 3(a) and 3(c). It is known from the inequality (1.36) that a B instability corresponds to frequency ranges for which the ordinates in Figs. 3(a) and 3(b) are of opposite signs. The quantity σ corresponding to the graph of Fig. 3(a) is chosen to be relatively small. Therefore, the P instabilities are considerably more intense, i.e., characterized by considerably larger rate of growth $|\text{Im}\delta|$ than the B instabilities. For other somewhat larger values of σ (which nevertheless satisfy the inequality $\sigma \ll 1$) the contrast between the strong P instabilities and the weak B instabilities may be less accentuated.

A classification similar to the one used for P instabilities can be applied to B instabilities. Thus, B instabilities may be superluminous, subluminous, or counterstreaming. It can be shown that a superluminous P instability represented by A_{sp} continues in the neighboring higher frequency range as a superluminous B instability. Similarly, the counterstreaming B instability represented by A_c continues into the neighboring lower frequency range as a counterstreaming B instability.

There are also regions of no instability shown in Fig. 3(c). These regions correspond to frequency ranges in which the ordinates of Figs. 3(a) and 3(b) are either both positive or both negative.

V. KINEMATIC BEHAVIOR OF PLASMA-BEAM INSTABILITIES

The analysis of the dispersion equation (1.4) was based on the assumption that k is real and the roots of this equation were expressed in the form $\omega = \tilde{\omega} + \delta$ where ω is real. An instability was shown to occur when $\text{Im}\delta < 0$ and the expressions (1.30) and (1.37) provided a formulation for the rate of growth of excited waves in terms of various parameters of the plasma-beam system. A further discussion on this subject resulted in determining certain characteristic properties of these instabilities. Several criteria have been established for distinguishing between the excited P and B waves and between instabilities which may be superluminous, subluminous, and counterstreaming.

The above analysis did not indicate, however, the kinematic behavior of the instabilities. Thus, it is not known whether the instabilities produced by a helical beam are convective or nonconvective. In other plasma-beam systems the kinematic behavior of various instabilities has been well explored. In several recent investigations it was shown that in the absence of the magnetic field $\mathbf{B}_0$, there are longitudinal and hybrid instabilities and both instabilities are convective.[17] In an analogous situation concerning the interaction of a beam with a stationary dielectric medium (comprising bound oscillators) it was shown that the resulting Bohr and Vavilov-Cherenkov instabilities are convective.[18]

The character of the plasma-beam interaction depends on the presence or absence of the static magnetic field $\mathbf{B}_0$. If $\mathbf{B}_0 = 0$, one has an instability when the velocity of the beam $c\beta$ is equal to the phase velocity ω/k of the excited wave.[4] Such an equality does not exist, however, when the plasma-beam system is immersed in a magnetic field $\mathbf{B}_0$. In such case the relationship between these two velocities is strongly dependent on the presence or absence of $\beta_\perp$. If $\beta_\perp = 0$ (linear beam), the velocity of the beam exceeds the phase velocity of the wave, i.e., there is a superluminous instability. It has been shown in a recent investigation that such a superluminous instability is convective.[7]

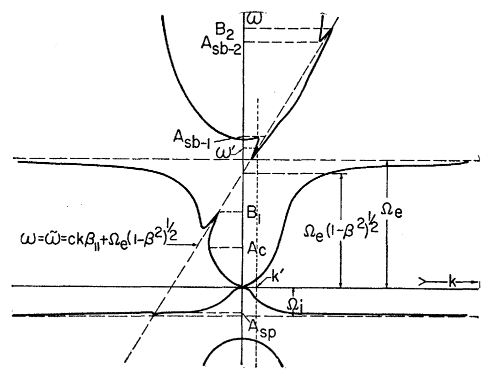


FIG. 11. "$\omega - k$" diagram for a helical electron beam interacting with a stationary plasma [case corresponding to Fig. 3(a)].

[17] J. E. Drummond and D. B. Chang, Bull. Am. Phys. Soc. **6**, 411 (1958); V. T. Kurilko, V. D. Shapiro, and IA. B. Fainberg, Report of Physico-Technical Institute of Ukrainian Academy of Sciences, UK. S.S.R., 1959 (unpublished); Jacob Neufeld and Harvel Wright, Phys. Rev. **127**, 346 (1962).

[18] Jacob Neufeld and Harvel Wright, Phys. Rev. **124**, 1 (1961).

On the other hand, if $\beta_{\perp}\neq 0$ (helical beam), there is a subluminous and a counterstreaming instability. Using a graphical analysis of the dispersion equation (1.4) it will be shown that the superluminous and subluminous instabilities are convective, whereas the counterstreaming instabilities are nonconvective.

The graphical representation of the dispersion equation (1.4) in the $\omega-k$ plane is shown in Figs. 11 and 12. The graphs shown in these two figures correspond to the plasma-beam systems of the type illustrated in Figs. 3(a) and 2, respectively.

The dispersion equation (1.4) has the form $F_p-\sigma F_b=0$, i.e., it contains the term F_p representing the stationary plasma and the term σF_b representing the beam. In examining the behavior of the graphs in Figs. 11 and 12, it will be useful to plot an auxiliary diagram as shown in Fig. 10. This auxiliary diagram shows a graph $F_p=0$ representing the dispersion in the unperturbed plasma and a graph $\omega=ck\beta_{\parallel}+\Omega_e(1-\beta^2)^{1/2}$ representing the resonance between the beam and the excited waves. One can observe in Fig. 10 that for any real value of k there are four real values of ω which satisfy the dispersion equation $F_p=0$. Of particular interest are the points of intersection of the line $\omega=ck\beta_{\parallel}+\Omega_e(1-\beta^2)^{1/2}$ with the graph of $F_p(k,\omega)=0$. These points of intersection shown in Fig. 10 represent waves which are in resonance with the beam and which at the same time can be propagated through the stationary plasma. These waves, when perturbed by the beam, become "excited P waves." The frequencies of these waves are labeled A_{sp}, A_c, A_{sb-1}, and A_{sb-2} on the frequency axis (ordinate). This labeling is to be compared with the labeling on the frequency axis (abscissa) of Fig. 3(a). An example of the actual values of the parameters η, A, $\beta_{\parallel}$, and $\beta_{\perp}$ for a plasma-beam system of this type can be found in Fig. 5(a) where $A=10$, $\beta_{\parallel}=0.7$, $\eta=1837$, and $\beta_{\perp}=0.5$.

By comparing Fig. 10 with Fig. 11 it can be seen how the diagram of the unperturbed plasma is modified by the interaction with the beam. For small values of σ the diagram for the stationary plasma is appreciably changed only in the immediate neighborhood of the line $\omega=\tilde{\omega}$, where $\tilde{\omega}=ck\beta_{\parallel}+\Omega_e(1-\beta^2)^{1/2}$.

Figure 11 should now be compared with Fig. 3(c). All points that are labeled on the frequency axis in Fig. 3(c) are correspondingly labeled on the frequency axis in Fig. 11. It is noted that certain regions exhibit no instability. For example, consider assigning a real value $k=k'$ such that $\tilde{\omega}'=ck'\beta_{\parallel}+\Omega_e(1-\beta^2)^{1/2}$ is between the points Ω_e and A_{sb-1}. Figure 3(c) then shows that there is no instability, i.e., ω is real. This can also be seen from Fig. 11. For the value $k=k'$ marked in Fig. 11, which is such that $\tilde{\omega}$ is between Ω_e and A_{sb-1}, there are six real values of ω which satisfy the dispersion equation $F_p-\sigma F_b=0$. These frequencies correspond to the intersections of the curve with a vertical line through the point $k=k'$. Regions of instability indicated in Fig. 3(c) can also be seen in Fig. 11 since for these values of k there are four real values of ω and thus two complex values.

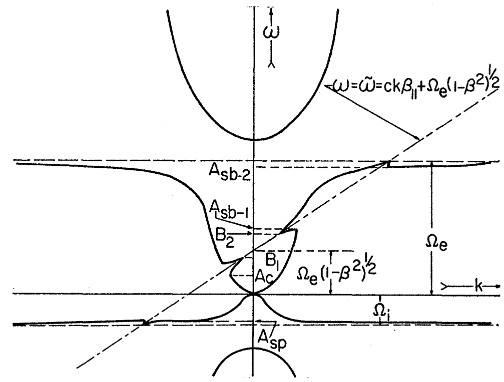


FIG. 12. "$\omega-k$" diagram for a helical electron beam interacting with a stationary plasma (case corresponding to Fig. 2).

Figure 12 is similar to Fig. 11 but illustrates a plasma-beam system of the type shown in Fig. 2 or shown, for example, in Fig. 5(b) for $A=100$ and $\beta_{\parallel}=0.05$.

Figure 12 is such that the line $\omega=\tilde{\omega}=ck\beta_{\parallel}+\Omega_e(1-\beta^2)^{1/2}$ representing the resonance between the beam and the excited waves intersects a different branch of the curve $F_p=0$.

Figures 11 and 12 can be used to determine the kinematic behavior of the plasma-beam systems. The procedure outlined by Sturrock[11] was described in detail for a diagram of a similar nature to those of Figs. 11 and 12 in a previous investigation.[7] It should be kept in mind that the resonance condition (1.11) limits the region of applicability of these figures to a narrow strip along the line $\omega=\tilde{\omega}$. By applying the above procedure, subject to this restriction, it is seen that the counterstreaming P instability is not convective but the superluminous and subluminous P instabilities are convective.